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Bayesian Estimation of Time To Test Transform for The Lomax Distribution Using Censored Sample under Different Loss Functions

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ABSTRACT
The concept of Time To Test Transform (TTT) is well known for its applications in different fields of study such as reliability analysis, econometrics, stochastic modeling and ordering distributions. In this article, we estimate the TTT for the Lomax function based on censored sample. The Bayes estimates are evaluated under squared error, entropy, precautionary loss functions. The empirical evaluation of the estimates is done using a simulation study.

KEYWORDS
Time To Test Transform; Censored sampling; Entropy loss function; Lomax distribution; Precautionary loss function; Prior distribution; Squared error loss function.

1. Introduction

In reliability and life testing, the important determinants are testing time and cost of sample units. To achieve reduction in testing time and cost of sample units, different censored sampling procedures are suggested. In the statistical literature, many researchers have concentrated on providing estimators of different parameters and parametric functions useful in reliability studies using different life distributions and under various censored sampling schemes. In general, censored sampling mechanism is to observe the complete life time of few experimental units out of n units.

The TTT-plot an empirical and scale invariant plot based on failure data, and the corresponding asymptotic curve, named the scaled TTT-Transform were introduced by [1] and used for model identification purposes. Since then these tools have proven to be very useful in several applications in reliability. The applications of this transform in econometrics and its close relationship with the Lorenz curve have been studied by many authors including [3], [4], [6], [5], among others.

Let $F(x)$ denote the life distribution of a certain type of units, i.e. $F(t)$ is the

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probability that the unit will fail before time 1. Furthermore, let $\bar{F}(x)$ be the corresponding survival function, i.e. the probability that the unit survives beyond x. The mean is denoted by μ and is calculated by integrating the survival function, i.e.

$$\mu = \int_0^\infty \bar{F}(x)dx$$

Although many of the results are true under more general conditions, we assume for simplicity that $F(t)$ is continuous and strictly increasing. This means for instance that the usual inverse function $F^{-1}(x)$ exists. The total time on test (TTT) defined as $T(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx$ for $0 \leq t \leq 1$.
With these notations in mind we can define the scaled TTT-transform as

$$\phi(t) = \frac{1}{\mu} \int_0^{F^{-1}(t)} \bar{F}(x)dx \quad \text{for } 0 \leq t \leq 1 \tag{1.1}$$

$$\bar{F} = 1 - F, \;\; \mu = \int_0^\infty \bar{F}(x)dx \text{ and } F^{-y} = infx : F(x) \geq y \text{ for } 0 \leq y \leq 1$$

Lomax distribution has been used as an alternative to the exponential, gamma and Weibull distributions for heavy tailed data by [2]. The Lomax distribution is considered as an important model of lifetime models since it belongs to the family of decreasing failure rate. Lomax distribution is one of the well known distributions that is very useful in many fields such as engineering and reliability and life testing. However, this distribution does not provide great flexibility in modeling data. Thus, Lomax distribution can be generalized by presenting additional parameters such as shape, scale or location in the distribution and then observing the characteristics of the new distribution. The probability density function of Lomax distribution is given by

$$f(x : \theta, \lambda) = \frac{\theta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\theta+1)} \qquad x, \theta, \lambda > 0 \tag{1.2}$$

where θ and λ are shape and scale parameters respectively.

For the above model,the TTT simplifies to

$$\phi(t) = 1 - (1-t)^{\frac{\theta-1}{\theta}} \tag{1.3}$$

After this brief introduction, we derive the likelihood function under the right censored model for Lomax distribution in Section 2. The Bayesian estimation of the TTT-transform of the Lomax distribution is discussed in Section 3. Finally in Section 4, we assess the performance of the estimates using Monte Carlo simulation.

2. Likelihood function

Consider a right censored sample $x_{(1)}, x_{(2)}, \ldots, x_{(n-s)}$ observations from a Lomax distribution. The likelihood function under the right censored model for Lomax distribution is given by

$$L(\underset{\sim}{x}|\theta, \lambda) = (1 - F_{n-s})^s \prod_{i=1}^{n-s} f(x_i)$$

$$= \left[1 + \frac{x_{n-s}}{\lambda}\right]^{-s\theta} \prod_{i=1}^{n-s} \frac{\theta}{\lambda} \left(1 + \frac{x_i}{\lambda}\right)^{(\theta+1)} \tag{2.1}$$

3. Bayesian Estimation

Bayesian methods have produced some remarkably efficient solutions to difficult estimation problems. Researchers often choose the techniques on practical grounds, rather than in adherence to their philosophical basis. Indeed, for some, the Bayesian estimator is merely an algorithm. The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than other. Here we suggest uniform prior, exponential prior and gamma prior for the Bayesian estimation when the scale parameter is known and the joint prior for the scale parameter is unknown.

3.1. When λ is known

3.1.1. Bayesian estimation of TTT under uniform prior by using different loss functions

The uniform prior for θ is given by,

$$g(\theta) \propto 1, \ \ \theta > 0 \tag{3.1}$$

Combining the prior distribution (3.1) and the likelihood function (2.1), the posterior density of θ is derived as follows

$$f(\theta|\underset{-}{x}) \propto \theta^{n-s} exp\{-\theta q\} \tag{3.2}$$

$$\text{where } q = s\, log\left(1 + \frac{x_{n-s}}{\lambda}\right) + \sum_{i=1}^{n-s} log\left(1 + \frac{x_i}{\lambda}\right) \tag{3.3}$$

Let ϕ be a parameter and replacing θ in (3.2) in terms of ϕ using (1.3), we get the posterior density of the TTT as

$$f(\phi|\underset{-}{x}) = \frac{R_\phi^{n-s+2}(1-\phi)^{-1} exp[-R_\phi q]}{O_1(t,0)} \tag{3.4}$$

$$\text{where } O_1(t,d) = \int_0^t \phi^d R - \phi^{n-s+2}(1-\phi)^{-1} exp[-R_\phi q] d\phi \tag{3.5}$$

In order to select the 'best' estimator, a loss function must be specified and is used to represent a penalty associated with each of the possible estimates. Squared error loss function (SELF) is a commonly used loss function and the Bayes estimator under the above loss function is given by

$$\widehat{\theta}_{CS} = E(\theta|\underset{-}{x})$$

The Bayes estimator of TTT under Squared error loss function is given by

$$\widehat{\phi}_{CS1} = E(\phi|x) = \frac{O_1(1)}{O_1(0)} \tag{3.6}$$

The entropy loss function (ELF) is a special case of general entropy loss function and the Bayes estimate under ELF is given by

$$\widehat{\theta}_{CE} = [E(\theta^{-1}|\underset{-}{x})]^{-1}$$

The Bayes estimator of TTT under Entropy Loss Function is given by

$$\widehat{\phi}_{CE1} = [E(\phi^{-1}|\underset{-}{x}]^{-1} = \frac{O_1(1)}{O_1(-1)} \tag{3.7}$$

The precautionary loss function (PLF) is an asymmetrical loss function and the Bayes estimate under PLF is given by

$$\widehat{\theta}_{CP} = \sqrt{E(\theta^2|\underline{x})}$$

The Bayes estimator of TTT under Precautionary Loss function is given by

$$\widehat{\phi}_{CP1} = \sqrt{E(\phi^2|\underline{x})} = \left[\frac{O_1(2)}{O_1(0)}\right]^{\frac{1}{2}} \tag{3.8}$$

3.1.2. Bayesian estimation of TTT under exponential prior by using different loss functions

In this section, we gives the Bayes estimators of TTT using exponential prior under different loss functions.

The exponential prior for θ is given by,

$$g(\theta) \propto exp\{-\theta\omega\}, \quad \theta, \omega > 0 \tag{3.9}$$

Combining the prior distribution (3.8) and the likelihood function (2.1), the posterior density of θ is derived as follows

$$f(\theta|\underline{x}) \propto \theta^{n-s} exp\{-\theta b\} \tag{3.10}$$

$$\text{where} \quad b = \omega + q \tag{3.11}$$

and q is given in (3.3).

Replacing θ in (3.10) in terms of ϕ using (1.3), we get the posterior density of the TTT as

$$f(\phi|\underline{x}) = \frac{R_\phi^{n-s+2}(1-\phi)^{-1} exp[-R_\phi b]}{O_2(t,0)} \tag{3.12}$$

$$\text{where} \quad O_2(t,d) = \int_0^t \phi^d R_\phi^{n-s+2}(1-\phi)^{-1} exp[-R_\phi b] d\phi \tag{3.13}$$

and R_ϕ is given in (3.5).

The Bayes estimator of TTT under Squared error loss function is given by

$$\widehat{\phi}_{CS2} = E(\phi|x) = \frac{O_2(1)}{O_2(0)} \tag{3.14}$$

The Bayes estimator of TTT under Entropy Loss Function is given by

$$\widehat{\phi}_{CE2} = [E(\phi^{-1}|\underline{x})]^{-1} = \frac{O_2(0)}{O_2(-1)} \tag{3.15}$$

The Bayes estimator of TTT under Precautionary Loss function is given by

$$\widehat{\phi}_{CP2} = \sqrt{E(\phi^2|\underline{x})} = \left[\frac{O_2(2)}{O_2(0)}\right]^{\frac{1}{2}} \tag{3.16}$$

3.1.3. Bayesian estimation of TTT under gamma prior by using different loss functions

In this section, we gives the Bayes estimators of TTT using gamma prior under different loss functions.

The gamma prior for θ is given by,

$$g(\theta) \propto \theta^{p-1} exp\{-\theta\tau\}, \quad \theta, p, \tau > 0 \tag{3.17}$$

Combining the prior distribution (3.17) and the likelihood function (2.1), the posterior density of θ is derived as follows

$$f(\theta|\underline{x}) \propto \theta^{L-1} exp\{-\theta g\} \tag{3.18}$$

$$\text{where} \quad L = n - s + p, \; g = \tau + q \text{ and } q \text{ is given in (3.3)} \tag{3.19}$$

Replacing θ in (3.18) in terms of ϕ using (1.3), we get the posterior density of the TTT as

$$f(\phi|\underline{x}) = \frac{R_\phi^{L+1}(1-\phi)^{-1} exp\left[-R_\phi g\right]}{O_3(t,0)} \tag{3.20}$$

$$\text{where } O_3(t,d) = \int_0^t \phi^d R_\phi^{L+1}(1-\phi)^{-1} exp\left[-R_\phi g\right] d\phi \tag{3.21}$$

and R_ϕ is given in (3.5).
The Bayes estimator of TTT under Squared error loss function is given by

$$\widehat{\phi}_{CS3} = E(\phi|x) = \frac{O_3(1)}{O_3(0)} \tag{3.22}$$

The Bayes estimator of TTT under Entropy Loss Function is given by

$$\widehat{\phi}_{CE3} = [E(\phi^{-1}|\underline{x})]^{-1} = \frac{O_3(0)}{O_3(-1)} \tag{3.23}$$

The Bayes estimator of TTT under Precautionary Loss function is given by

$$\widehat{\phi}_{CP3} = \sqrt{E(\phi^2|\underline{x})} = \left[\frac{O_3(2)}{O_3(0)}\right]^{\frac{1}{2}} \tag{3.24}$$

3.2. When λ is unknown

In this section, we gives the Bayes estimators of TTT using the joint prior under different loss functions.

The joint prior for θ is given by,

$$g(\theta,\lambda) \propto \frac{\theta^{p-1} exp\{-\theta\tau\}}{\lambda}, \quad \theta, p, \lambda > 0 \tag{3.25}$$

Combining the prior distribution (3.25) and the likelihood function (2.1), the posterior density of θ is derived as follows

$$f(\theta|\underline{x}) \propto \theta^{L-1} \int_0^\infty exp\{-(\theta g + e)\} d\lambda \tag{3.26}$$

$$\text{where } e = \sum_{i=1}^{n-s} log\left(1+\frac{x_i}{\lambda}\right) + (n-s)log(\lambda) \text{ and } L,\ g \text{ are given in (3.19)} \tag{3.27}$$

Replacing θ in (3.26) in terms of ϕ by that (1.3), we get the posterior density of the TTT as

$$f(\phi|\underline{x}) = \frac{R_\phi^{L+1}(1-\phi)^{-1}\int_0^\infty exp\{-(\theta g + e)\}d\lambda}{O_4(t,0)} \tag{3.28}$$

$$\text{where } O_4(t,d) = \int_0^t \int_0^\infty \phi^d R_\phi^{L+1}(1-\phi)^{-1} exp[-(R_\phi g + e)] d\lambda d\phi \tag{3.29}$$

and R_ϕ is given in (3.5).
0.1cm] The Bayes estimator of TTT under Squared error loss function is given by

$$\widehat{\phi}_{CS4} = E(\phi|x) = \frac{O_4(1)}{O_4(0)} \tag{3.30}$$

The Bayes estimator of TTT under Entropy Loss Function is given by

$$\widehat{\phi}_{CE4} = [E(\phi^{-1}|\underline{x})]^{-1} = \frac{O_4(0)}{O_4(-1)} \tag{3.31}$$

The Bayes estimator of TTT under Precautionary Loss function is given by

$$\widehat{\phi}_{CP4} = \sqrt{E(\phi^2|\underline{x})} = \left[\frac{O_4(2)}{O_4(0)}\right]^{\frac{1}{2}} \tag{3.32}$$

It may be noted that the Bayes estimator of TTT under three loss functions are not reducible in closed form. Hence, we seek suitable numerical integration to obtain estimates.

4. Simulation Study

In the absence of real data, we study the performance of the estimators obtained so far using simulated data. With different values of the parameters of the model, right censored samples of different sizes are generated and we compare the bias and the mean square errors of TTT. Integration is carried out using Cubature-package of R for the evaluation of estimates. A Monte Carlo simulation has been carried out for establishing the performance of the estimates. In the first step, we generate samples of sizes 25, 50 and 75 with censoring of 10%. The above measures are calculated empirically using 1,000 Monte Carlo runs for different choices of the parameters. The bias and MSEs of the different estimators are given in table.

Table 1
True value of TTT when t=0.3

θ	2.5	3	3.5	4	4.5
trueϕ	0.19266	0.21163	0.22490	0.23471	0.24226

Table 2
Bias and MSEs (in parentheses) of the estimates of TTT under uniform prior

n	θ	$\widehat{\phi}_{CS1}$	$\widehat{\phi}_{CE1}$	$\widehat{\phi}_{CP1}$
25	2.5	0.12180	0.13250	0.12580
		(0.00780)	(0.00870)	(0.01452)
	3	0.13740	0.12897	0.11568
		(0.54620)	(0.04251)	(0.01254)
	3.5	0.13651	0.12808	0.11479
		(0.23560)	(0.36250)	(0.01410)
	4	0.14530	0.13687	0.12358
		(0.00020)	(0.00120)	(0.00320)
	4.5	0.14986	0.14143	0.12814
		(0.00630)	(0.00240)	(0.00560)
10	2.5	0.12071	0.13141	0.12471
		(0.00889)	(0.00979)	(0.01561)
	3	0.13634	0.12791	0.11462
		(0.02430)	(0.03192)	(0.05450)
	3.5	0.12772	0.11929	0.10600
		(0.23580)	(0.36370)	(0.01730)
	4	0.14441	0.13598	0.12269
		(0.00160)	(0.00960)	(0.02560)
	4.5	0.14530	0.13687	0.12358
		(0.26010)	(0.39562)	(0.07180)
75	2.5	0.11985	0.13055	0.12385
		(0.03319)	(0.04171)	(0.07011)
	3	0.13548	0.12705	0.11376
		(0.02590)	(0.04152)	(0.08010)
	3.5	0.12686	0.11843	0.10514
		(0.02450)	(0.03312)	(0.05770)
	4	0.14355	0.13512	0.12183
		(0.00074)	(0.00874)	(0.02474)
	4.5	0.14444	0.13601	0.12272
		(0.01280)	(0.07680)	(0.20480)

Table 3
Bias and MSEs (in parentheses) of the estimates of TTT under exponential prior

n, s	θ	$\widehat{\phi}_{CS2}$	$\widehat{\phi}_{CE2}$	$\widehat{\phi}_{CP2}$
25	2.5	0.13780	0.11270	0.11568
		(0.02546)	(0.02545)	(0.12520)
	3	0.13456	0.13245	0.13589
		(0.25860)	(0.68450)	(0.00254)
	3.5	0.13370	0.13159	0.13503
		(0.00250)	(0.00020)	(0.01240)
	4	0.13456	0.13245	0.13579
		(0.00040)	(0.00240)	(0.00030)
	4.5	0.12258	0.11245	0.14865
		(0.00960)	(0.00870)	(0.00420)
10	2.5	0.13671	0.11161	0.11459
		(0.02655)	(0.02654)	(0.12629)
	3	0.13350	0.13139	0.13483
		(0.07637)	(0.17610)	(0.27560)
	3.5	0.12491	0.12280	0.12624
		(0.00290)	(0.00260)	(0.01270)
	4	0.13367	0.13156	0.13490
		(0.00320)	(0.01920)	(0.00240)
	4.5	0.11802	0.10789	0.14409
		(0.07927)	(0.00054)	(0.28830)
75	2.5	0.13585	0.11075	0.11373
		(0.10292)	(0.20264)	(0.00504)
	3	0.13264	0.13053	0.13397
		(0.07957)	(0.19530)	(0.27800)
	3.5	0.12406	0.12195	0.12539
		(0.07677)	(0.17850)	(0.27590)
	4	0.13281	0.13070	0.13404
		(0.00234)	(0.01834)	(0.00154)
	4.5	0.11716	0.10703	0.14323
		(0.02560)	(0.15360)	(0.01920)

Table 4
Bias and MSEs (in parentheses) of the estimates of TTT under gamma prior

n, s	θ	$\widehat{\phi}_{CS3}$	$\widehat{\phi}_{CE3}$	$\widehat{\phi}_{CP3}$
25	2.5	0.13540	0.11250	0.11841)
		(0.02520)	(0.05100)	(0.00050
	3	0.13890	0.13740	0.12546
		(0.06400)	(0.06020)	(0.00547)
	3.5	0.13804	0.13654	0.12460
		(0.00021)	(0.23650)	(0.00140)
	4	0.13880	0.17451	0.11587
		(0.04010)	(0.00256)	(0.00244)
	4.5	0.13854	0.12856	0.12856
		(0.00650)	(0.00450)	(0.00080)
10	2.5	0.13431	0.11141	0.11731
		(0.02629)	(0.05209)	(0.00159)
	3	0.13784	0.13634	0.12440
		(0.10140)	(0.10250)	(0.00100)
	3.5	0.12925	0.12775	0.11581
		(0.04031)	(0.01245)	(0.00380)
	4	0.13794	0.17365	0.11501
		(0.32080)	(0.02048)	(0.01920)
	4.5	0.13398	0.12451	0.12770
		(0.14171)	(0.06050)	(0.00480)
75	2.5	0.13345	0.11055	0.11645
		(0.12769)	(0.15459)	(0.00259)
	3	0.13698	0.13548	0.12354
		(0.42220)	(0.12298)	(0.02027)
	3.5	0.12840	0.12690	0.11496
		(0.14150)	(0.10506)	(0.00340)
	4	0.13709	0.17280	0.11416
		(0.31994)	(0.01962)	(0.01834)
	4.5	0.13312	0.12365	0.12685
		(0.00045)	(0.16384)	(0.15360)

Table 5
Bias and MSEs (in parentheses) of the estimates of TTT when λ is unknown

n, s	θ	$\widehat{\phi}_{CS4}$	$\widehat{\phi}_{CE4}$	$\widehat{\phi}_{CP4}$
25	2.5	0.09991	0.11454	0.10395
		(0.06242)	(0.06965)	(0.11622)
	3	0.09062	0.10094	0.08994
		(0.01702)	(0.10205)	(0.27205)
	3.5	0.11402	0.12999	0.11139
		(0.00792)	(0.01208)	(0.03126)
	4	0.10692	0.11597	0.09692
		(0.45245)	(0.50465)	(0.84222)
	4.5	0.08285	0.09174	0.07391
		(0.05822)	(0.02795)	(0.05934)
10	2.5	0.09832	0.10904	0.10234
		(0.06235)	(0.06955)	(0.11615)
	3	0.08999	0.09904	0.08634
		(0.01608)	(0.10106)	(0.27105)
	3.5	0.11317	0.12295	0.11045
		(0.00787)	(0.01194)	(0.03111)
	4	0.10527	0.11415	0.09581
		(0.45144)	(0.50365)	(0.84125)
	4.5	0.08195	0.09081	0.07257
		(0.05819)	(0.02781)	(0.05928)
75	2.5	0.09785	0.10790	0.10207
		(0.06132)	(0.06850)	(0.11518)
	3	0.08860	0.09845	0.08597
		(0.01500)	(0.10004)	(0.27008)
	3.5	0.11111	0.12144	0.10949
		(0.00684)	(0.01090)	(0.03017)
	4	0.10425	0.11315	0.09485
		(0.45144)	(0.50365)	(0.84125)
	4.5	0.08041	0.08912	0.07247
		(0.06221)	(0.06965)	(0.11628)

In this paper, Bayesian method is used for estimating the TTT transform of Lomax distribution based on right censored samples. It has been noticed, from the tables, that the bias decreases as the sample size increases.

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